BRIDGING THE VECTOR (CALCULUS) GAP

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There is a surprisingly large gap between the way mathematicians on the one hand, and physical scientists and engineers on the other, think about vectors. This gap is particularly apparent when students attempt to apply the vector calculus they have learned in mathematics courses to the problems in electromagnetism which they encounter in subsequent physics or engineering courses. We are developing materials in an effort to bridge this "vector calculus gap". Our approach suggests changes which could be made in the teaching of related material, some of which is taught at the high school level, the most important example being the basic properties of vectors. Here, we discuss this even more fundamental "vector gap", and ways to bridge it.

Introduction

Each of us has taught vector calculus for many years: one of us (TD) as part of the second year calculus sequence in the mathematics department, and the other (CAM) as part of the third year physics course on mathematical methods. The latter course officially has the former as a prerequisite, although there is considerable overlap in the material covered. In fact, part of the reason the physics course exists is to teach physics majors the "right" way to use this material, the sense being that the mathematics department has failed to do so. Nevertheless, it literally took us years to discover how little resemblance this subject as taught by mathematicians bears to the version taught by physicists.

This "vector calculus gap" [1] is particularly apparent when students attempt to apply the vector calculus they have learned in mathematics courses to the problems in electromagnetism which they encounter in subsequent physics or engineering courses. In a recent paper [2], we proposed a new way to approach some problems in vector calculus, in an effort to bridge this gap. It is instructive to contrast the treatment of this material in mathematics and physics textbooks. An excellent presentation of the traditional mathematics approach can be found in Stewart [3], while the physics approach is beautifully described in Griffiths [4]. Our work has been strongly influenced by previous efforts to "bridge the gap", including the classic description by Schey [5] of the physics view of divergence and curl, some of which (and more) has been thoughtfully incorporated in the more recent multivariable calculus text by the Calculus Consortium [6].

Our experience teaching vector calculus leads us to emphasize geometric reasoning over algebraic computation when working with vectors. While many of our incoming students are fairly comfortable with algebraic manipulations of vectors, few of them have any experience with the underlying geometry. This "vector gap" is a serious handicap for students of physics and engineering. We believe that students should be exposed to the geometry of vectors right from the beginning. In this paper, we make some specific suggestions about how this might be achieved.

One feature of our approach to vector calculus is the use of vector differentials, as discussed in [2]. After first discussing our geometric approach to vectors, we then indicate briefly how differentials can be used in differential and integral calculus, providing a natural link to our subsequent use of differentials in vector calculus.

Vectors

What is a vector? Most textbooks define vectors to be pairs or triples of numbers. This algebraic definition obscures the geometry: Vectors are arrows in space, possessing magnitude and direction. Yes, of course, it is easy to show that the traditional, basis-dependent definition of a vector in fact defines something geometric, that is, basis-independent. But why not start with the geometry? Teaching students that vectors are "really" just an ordered list of numbers not only obscures the geometry, but also makes it hard to use some other basis.

When we give the components of a vector, we therefore insist on writing the basis explicitly. Thus, we write $\vec{\boldsymbol{v}} = 3\,\hat{\boldsymbol{\imath}} + 4\,\hat{\boldsymbol{\jmath}}$, not $\vec{\boldsymbol{v}} = \langle 3, 4 \rangle$. (We denote the unit vectors in the x and y directions by $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$, respectively. Many other notations are in common use.) Students tend to forget that the basis is there unless they can see it — leading them to think of vectors as algebraic objects rather than geometric objects.

Why would one ever need to use a different basis? Consider the problem of relating magnetic north to "true" north. This is a daily fact of life for airplane pilots, and the maps used by pilots indicate both of these directions. We often use a 2-dimensional version of this problem as an introduction to the use of different bases; it is hard even to formulate this problem without explicitly referring to both bases.

But there is an even more important reason to introduce other bases. We don't think twice about using spherical coordinates to solve problems with spherical symmetry. It is equally natural to use an orthonormal basis adapted to the symmetry. For example, in polar coordinates (r, ϕ) in the plane, it is natural to introduce the unit vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ which point in the direction of increasing r and ϕ, respectively. Physicists and engineers do this all the time! Yet most mathematicians have never seen this notation, much less used it. (We choose ϕ for the polar angle in order to agree with the standard conventions for spherical coordinates used by everyone but mathematicians; for further discussion of this issue, see [7].)

It is worth noting that the use of vector differentials, which we advocate in [2], extends naturally to these adapted bases, especially for problems having symmetry. This greatly enhances the flexibility of this approach, and in particular makes it easy to handle the highly symmetric surfaces fundamental to electromagnetism.

Group Activities

There are (at least) 3 parts to the scientific process:

- Determining what the problem is;
- Deciding which strategy to adopt;
- Conveying the results to others.

We typically devote one class meeting per week to small group activities. Such activities often work best if each person in the group has a particular responsibility or task, such as the ones above. For instance, the *task master* determines what the problem is, and suggests how to go about it. The *cynic* questions whether the right approach is being used, thus ensuring that the best strategy is adopted. The *recorder* keeps track of what is being done, and the *reporter* conveys it to others.

At the first group activity, we provide students with a description of these roles, along the following lines:

Small group activities often work best if each person in the group has a particular responsibility or task. One of the most important outcomes of small group activities is that you will clarify your own ideas by discussing concepts. Teachers tend to learn more than students! In addition, you will learn to communicate these ideas to others — a vital skill for the workplace! Small group activities are not a competition to see which group can get done first. If the group moves on without a member, everyone loses.

Task Master: *The responsibility of the task master is to ensure that the group completes all of the parts of the work.*

"Part 1 says that we must ... . How shall we do it?"

"What you had for lunch doesn't seem relevant. Can we get back to the main question?"

Cynic: *The responsibility of the cynic is to question everything the group does and to ensure that everyone in the group understands what is going on.*

"Why are we doing it this way?"

"Wouldn't it be better if we did it this other way?"

"I don't understand this part of what we are doing, let's go over it again."

Recorder: *The responsibility of the recorder is to record the group's answers.*

"Do we agree that the answer to Part 3 is ... ?"

"I have written ... for Part 2. Is that what we want to say?"

Reporter: *The responsibility of the reporter is to report to the class as a whole.*

(Since everyone in the group needs to be accountable for the material, the reporter may not be designated at the beginning of the activity. Be prepared!)

The ability to work smoothly in a group is highly prized in industry — and usually requires training. These roles can certainly overlap; the roles of recorder and reporter, for instance, are often combined. In a smoothly running group, with everybody participating, there is of course no need to force people back into narrow roles. But some guidance is usually necessary in order to reach that goal.

Groups of 3 probably work best — everyone is assigned an essential role, and it is hard to make forward progress without everyone contributing. Larger groups make it too easy for 1 or more members to remain silent, either because they don't understand or because they'd rather work on their own.

We have found it to be essential that the instructor enforce the roles. Roles should be assigned; don't let students choose. For that matter, groups should be assigned; don't let friends choose the same groups. Rotate both periodically, the roles each week, and the groups every few weeks.

More importantly, the instructor should refer to the roles when communicating with the groups. Ask the recorder to explain what they've done so far — and insist that the explanation contain correct mathematics and correct use of language. If there's a problem, ask the task master to explain what the group is trying to do, and ask the cynic whether he/she is comfortable with the approach the group is using. In each case, make a point of asking who is playing which role, and of dealing with the group through the appropriate person.

Ideally, the recorder would be the only one taking notes until the group has solved the problem, after which, in this context, each student would write up the results individually based on the recorder's notes. In practice, that doesn't work very well. A better solution is simply to let everyone write down what they want, but until the group is done the instructor should really only pay attention to what the recorder has written.

It is a good idea to have the reporter present (partial) results to the class as a whole. While it is natural to choose the recorder to serve as reporter, this runs the risk that nobody else will be prepared. It's probably better not to choose a reporter in advance, but warn students that anybody might get chosen.

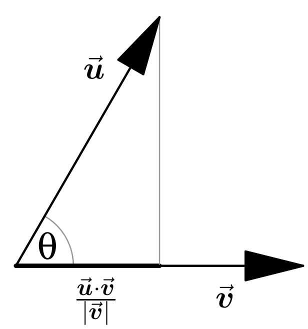
Sample Group Activity

A sample group activity appears below. This particular activity is often used at the beginning of a multivariable or vector calculus course to "review" the geometry of the dot product.

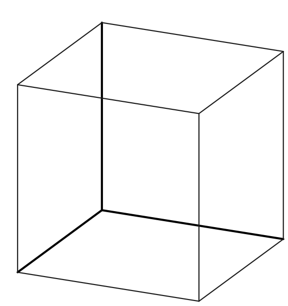
The *dot product* of 2 vectors $\vec{u}$ and $\vec{v}$ is just the (suitably scaled) projection of one vector along the other, as shown in the diagram at the right. It can be shown that this leads to the formula

$$\boxed{\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta}$$

where $|\vec{u}|$ is the *length* of $\vec{u}$, and θ the angle between $\vec{u}$ and $\vec{v}$. A *unit vector* is a vector whose length is 1. The unit vectors in the x, y, and z directions are usually denoted $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, respectively.



- Determine the dot products $\hat{\imath} \cdot \hat{\imath}$ and $\hat{\imath} \cdot \hat{\jmath}$.
- Determine the length of the diagonal of a unit cube.
- Determine the angle between the diagonal of a cube and an adjacent edge.



One important feature of this activity is that only the geometric definition of the dot product is given, not the more familiar algebraic formula in terms of components. This emphasizes that the dot product is geometric; components are not needed. In fact, this lab then turns the tables by using the geometric definition to *derive* the algebraic one. (Purists will point out that it is necessary to assume linearity to complete this argument.)

Another feature of this activity is the explicit use of basis vectors, such as $\hat{\boldsymbol{\imath}}$. As already noted, we *always* write vectors in the form $3\,\hat{\boldsymbol{\imath}} + 4\,\hat{\boldsymbol{\jmath}}$, rather than $\langle 3, 4 \rangle$. The latter notation hides the basis entirely, which makes it difficult even to write down a problem which uses more than one set of basis vectors. Note also the use of hats to denote unit vectors — a common practice in physics and engineering.

But the most important feature of this activity is the interplay between the various representations of the dot product. One needs *both* the geometric formula *and* the ability to compute dot products — essentially the algebraic formula — in order to complete the activity. Being able to play one representation off against another requires considerable mastery of both.

It is important that such group activities not be isolated from the rest of the course. This particular activity would likely be preceded by a lecture on the geometric definition of the dot product. This would be the appropriate place to at least mention, and possibly derive, the crucial fact that the dot product is linear (that is, that it distributes over addition). This activity should then be followed by a discussion about the interplay between different representations. In fact, the algebraic formula itself could be derived as part of this followup discussion! It would also be appropriate to assign some related homework problems.

Differentials

A key ingredient to our approach to vector calculus is to make differentials fundamental [2]. We acknowledge that mathematics *faculty* often have difficulty dealing with differentials, typically getting bogged down in asking just what, precisely, they are. While we will not pursue this here, we wish to emphasize that an approach based on differentials closely reflects the way most scientists and engineers successfully use calculus.

We also feel obliged to comment that the standard picture found in most calculus texts, in which dx is identified with Δx but dy is clearly different from Δy, is at best misleading and at worst flat out wrong. There is nothing in the concept of differential which depends on the choice of independent variables — that's the whole point of using them. The error is of course small, and linear approximations are useful. But it's a shame to demote this powerful tool to applications involving linearization.

Given a function $f = f(x)$, the *differential* of f is defined to be

$$df = \frac{df}{dx}\,dx$$

This definition is not very satisfying. What, after all, is dx? There are several ways to make this precise. We can think of dx as an infinitesimal change in the x direction, and df as the corresponding infinitesimal change in f. Then the above equation shows how these infinitesimal changes are related. That's exactly what a derivative is, a ratio of infinitesimal changes!

A better definition for our purposes, albeit informal, might be that dx and df are things which you integrate. Most likely, you are already used to using differentials — just think about substituting $u = x^2$ into the following integral:

$$\int 2x \sin(x^2)\, dx = \int \sin(u)\, du$$

Didn't you use the relation $du = 2x\, dx$? Those are differentials!

Another way to think of differentials is as the numerators of derivatives, in the sense that "dividing" differentials leads to derivatives. For instance, if $u = x^2$ as above, and $x = \cos\theta$, then

$$du = 2x\, dx = 2x(-\sin\theta\, d\theta) = -2\cos\theta \sin\theta\, d\theta$$

and "dividing" through by $d\theta$ yields

$$\frac{du}{d\theta} = -2\cos\theta\sin\theta$$

which is just the chain rule, since $u = x^2 = \cos^2\theta$. In fact, using differentials makes the chain rule automatic! Furthermore, these ideas generalize naturally to functions of several variables.

Summary

Our experience teaching vector calculus leads us to make the following strong recommendations to those teaching students about vectors:

- Use explicit basis vectors, such as $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$. That is, write $3\,\hat{\boldsymbol{\imath}} + 4\,\hat{\boldsymbol{\jmath}}$, not $\langle 3, 4\rangle$. It may not matter in your class, but it makes a difference later on.
- Emphasize that $|\vec{\boldsymbol{v}}|^2 = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}$, rather than making the Pythagorean theorem the starting point for finding the length of a vector.
- Emphasize that $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = |\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}|\cos\theta$. (Dot products are projections.)
- Emphasize that $|\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}| = |\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}|\sin\theta$. (Cross products are areas.)
- Emphasize that $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = 0 \Longleftrightarrow \vec{\boldsymbol{u}} \perp \vec{\boldsymbol{v}}$.
- Emphasize that $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} = \vec{0} \Longleftrightarrow \vec{\boldsymbol{u}} \parallel \vec{\boldsymbol{v}}$.

In addition to these recommendations, we also encourage the use of differentials in both differential and integral calculus, noting in particular how easy this makes both the chain rule and its inverse, substitution in integrals. We reiterate that this use of differentials closely reflects the way most scientists and engineers actually think about calculus

Related Projects

The ideas discussed here grew out of two curriculum reform projects at Oregon State University: the Vector Calculus Bridge Project, focusing primarily on second-year calculus and the Paradigms Project, a major reform of the upper-division physics curriculum. Further information about these projects can be found on their websites, at `http://www.physics.orst.edu/bridge` and `http://www.physics.orst.edu/paradigms`, respectively.

The Bridge Project is currently seeking beta testers. If you are interested in trying our materials in the classroom, please contact us.

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Bios

Tevian Dray received his BS in Mathematics from MIT in 1976, his PhD in Mathematics from Berkeley in 1981, spent several years as a physics postdoc, and is now Professor of Mathematics at Oregon State University. He considers himself a mathematician, but isn't sure. (Neither is his department.) Most of his research has involved general relativity. He is the Principal Investigator (PI) on the Vector Calculus Bridge Project.

Corinne Manogue received her AB in Mathematics and Physics from Mount Holyoke College in 1977, her PhD in Physics from the University of Texas in 1984, and is now Professor of Physics at Oregon State University. She continues to be amazed to find herself a physicist. Most of her research has been related to quantum gravity and superstring theory. She is the co-PI on the Vector Calculus Bridge Project and the PI on the Paradigms Project.

Corinne and Tevian have collaborated on many projects, including 2 children. In addition to their work on curriculum reform, they are trying to give a unified description of the fundamental particles of nature in terms of the octonions. They are currently on sabbatical at Mount Holyoke College, where they are the Hutchcroft Visiting Professors of Mathematics.

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